# Calculation of Bounds on Variables Satisfying Nonlinear Inequality Constraints

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**Abstract.** Existing techniques for solving nonconvex programming problems often rely on the availability of lower and upper bounds on the problem variables. This paper develops a method for obtaining these bounds when not all of them are available *a priori*. The method is a generalization of the method of Fourier which finds bounds on variables satisfying linear inequality constraints. First, nonlinear inequality constraints are converted to equivalent sets of separable constraints. Generalized variable elimination techniques are used to reduce these to constraints in one variable. Bounds on that variable are obtained and an inductive process yields bounds on the others.

**Key words.** Nonconvex, Fourier, global minimization.

## 1. Introduction

In their seminal paper, Falk and Soland [4] presented a method for finding a global solution to the problem of minimizing a separable nonlinear function subject to not necessarily linear constraints and bounded variables. This was extended by Soland [35]. McCormick [25] used the same general branch and bound approach to solve *factorable* optimization problems. In this section, we mention other literature on solving nonconvex programming problems where bounds on problem variables and/or the value of the objective function need to be provided.

There is very little written on solving nonconvex programming problems using the Falk-Soland branch and bound point of view. The dissertation by Leaver [21] used similar ideas, but gave a different method for computing concave over-estimating functions. Sisser's [33, 34] interval arithmetic techniques were used in conjunction with factorable functions to solve nonconvex problems. Both authors assumed that bounds on the variables were available.

In the most comprehensive recent survey [16] there are no other papers listed which deal with this approach. And, to the authors' knowledge, there is no work past or current which tries to find bounds so that the above nonconvex programming algorithms can be applied.

In the following we cite some of the papers surveyed that involved the requirement of bounds on problem variables and/or objective function value. The

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presentation is classified according to type of problem and/or technique used.

1. *Concave Minimization*: Benson [2]; Hoffman [12]; Horst [13, 14, 15, 18]; Rosen [31]; and Taha [36].
2. *Difference of convex programming*: Thach [37]; Thoai [39]; and Tuy [40, 41, 42].
3. *Reverse convex programming*: Avriel [1]; Hillested [10, 11]; Sen [32] and Tuy [40].
4. *Use of cutting planes, branch and bound and/or convex envelope techniques*: Benson [3], Falk [4], Gabot [7], Horst [13, 14, 18], McCormick [25, 27], Soland [35].

The organization of this paper is as follows. Section 2 characterizes functions and situations in which one can analyze the global behavior of the sum of two or more functions of one variable. In Section 3, techniques are introduced to find bounds on single variables from a set of inequality constraints. Section 4 develops a method for finding bounds on variables from a system of separable nonlinear constraints. This is a generalization of the method of Fourier [6] which applied to linear constraints only. The paper concludes with a summary and conclusion in Section 5.

## 2. Analyzing the Global Behavior of the Sum of Functions of One Variable

We characterize functions and situations in which one can analyze the global behavior of the sum of two or more functions of the same variable. The application for this will be given in Section 4.5 where Fourier's ideas for manipulating linear inequality constraints to find bounds on the variables are generalized to separable nonlinear inequality constraints. Essentially the idea is to eliminate variables from a sequence of inequality constraints until we are left with one or more inequality constraints in a single variable. From this, bounds on the single variable are obtained and a backward process yields bounds on all the variables.

The general problem considered is stated simply as: given two continuous functions $f_1(x)$ and $f_2(x)$ of the single variable $x$, find a positive scalar $\alpha$ and a scalar $\beta$ such that: $\alpha f_1(x) + \beta \geqslant f_2(x)$, for all $-\infty < x < +\infty$.

In general, if one pair $(\alpha, \beta)$ can be found there are an infinite number of such pairs. In general, for any $\alpha$, the $\beta$ is picked so that equality holds for at least one point on the real line. The problem of which of the infinite number of pairs to use is not considered here. There is no adequate measure for this since, in general, the functions tend to $\pm\infty$ as $|x| \to \infty$. The purpose for which a pair is used is to 'eliminate' a variable from a system of nonlinear separable constraints (Section 4.5). For this purpose, any pair will suffice.

Two different methods are devised to do this. Sometimes a lower or upper bound is available on the variable $x$. The methods presented here can be easily modified to handle these cases. For example if $x \geqslant L$ is known, all the operations which involve finding values for all $x$ s.t. $-\infty < x < +\infty$ are modified to all $x$ s.t. $L \leqslant x < +\infty$.

## 2.1. CALCULATING $\alpha$ AND $\beta$ FOR THE GENERAL FORM

*The first method* for finding $\alpha$ and $\beta$ such that

$$\alpha f_1(x) + \beta \geqslant f_2(x) \quad \text{for all } -\infty < x < +\infty$$

has two cases.

*Case I.*

$$\sup_{-\infty<x<+\infty} f_2(x) = \bar{f}_2 < \infty .$$

Here set $\alpha = 0$ and $\beta \geqslant \bar{f}_2$.

*Case II.*

$$\sup_{-\infty<x<+\infty} f_2(x) = +\infty .$$

The asymptotic behavior of $f_1(x)$ and $f_2(x)$ is analyzed as $|x| \to \infty$. From this analysis a value of $\alpha$ is obtained if one is available. Then a value of $\beta$ is gotten so that the desired inequality is satisfied for 'finite' points.

PART ONE. (Finding $\alpha$). Define

$$\eta_1 = \liminf_{\substack{-\infty<x<+\infty \\ f_2(x)\to+\infty}} \{f_1(x)/f_2(x)\} .$$

If there are values of $(\alpha, \beta)$, where $\alpha > 0$ such that the desired inequality is satisfied for all $x$, then

$$\alpha[f_1(x)/f_2(x)] + \beta/f_2(x) \geqslant 1$$

must hold for all $x$ where $f_2(x) > 0$. It follows from this that

$$\alpha\left\{\liminf_{\substack{-\infty<x<+\infty \\ f_2(x)\to+\infty}} [f_1(x)/f_2(x)]\right\} \geqslant 1 .$$

If $\eta_1 \leqslant 0$, there is no value of $\alpha > 0$ such that the desired inequality is satisfied. Intuitively this means $f_2(x)$ goes to plus infinity faster than $f_1(x)$ does along a sequence of points. (Or it means $f_1(x)$ goes to a value less than infinity along a sequence of points).

If $\eta_1 = \infty$, any value of $\alpha > 0$ will suffice. This corresponds to the case where for every sequence of points where $f_2(x)$ goes to $+\infty$, $f_1(x)$ goes to $+ = \infty$ faster.

Finally, if $\eta_1 > 0$, dividing through above by $\eta_1$ yields that $[\alpha \geqslant 1/\eta_1]$ must hold.

Sometimes an upper bound on $\alpha$ can be obtained. It is clear from the basic inequality that

$$\alpha[f_1(x)/f_2(x)] + \beta/f_2(x) \leqslant 1$$

must hold for all $x$ where $f_2(x) < 0$. From this it follows that

$$\alpha\eta_2 \leqslant 1$$

where

$$\eta_2 = \limsup_{\substack{-\infty < x < +\infty \\ f_2(x) \to -\infty}} [f_1(x)/f_2(x)] .$$

If $\eta_2 = \infty$, no value of $\alpha$ will suffice since $f_1(x) \to -\infty$ faster than $f_2(x) \to -\infty$ along a sequence of points. If $\eta_2 \leqslant 0$, any value of $\alpha > 0$ will do. Otherwise, $\alpha \leqslant 1/\eta_2$ must hold.

From the discussion above the following rules for setting $\alpha$ are derived:

- If both $\eta_1$ and $\eta_2$ are finite positive values, set $\alpha = [1/\eta_1 + 1/\eta_2]/2$.
- If $\eta_1 \leqslant 0$, or if $\eta_2 = \infty$, there are no values $(\alpha, \beta)$ such that the desired inequality is satisfied for all $x$.
- If $\eta_1 = \infty$, and $\eta_2 \leqslant 0$, set $\alpha = 1$, since any value of $\alpha > 0$ will handle the asymptotic cases.
- If $\eta_1 = \infty$, and $0 < \eta_2 < \infty$, set $\alpha = \eta_2/2$ since only the situation where $f_1$ and $f_2$ go to minus infinity are of concern.
- Finally, if $0 < \eta_1 < \infty$, and $\eta_2 \leqslant 0$, set $\alpha = 2/\eta_1$ since only the situation where both $f_1$ and $f_2$ go to plus infinity is of concern.

This way of finding a value of $\alpha$ allows for all the possibilities of the function $f_2(x)$ tending to $\pm\infty$.

PART TWO. (Finding $\beta$). Using the value of $\alpha$ gotten above, find a lower bound on

$$\inf_{-\infty < x < \infty} [\alpha f_1(x) - f_2(x)] .$$

If this is finite, set $\beta$ equal to any number higher than minus this value. Clearly, the values of $\alpha$ and $\beta$ obtained are those desired. The following theorem states the situation when the infimum is $-\infty$.

THEOREM. *If the infimum above is* $-\infty$, *there are no values of* $\alpha > 0$ *and associated* $\beta$ *such that the desired inequality holds for all* $x$.

*Proof.* If the infimum is $-\infty$, that means there is a sequence of points $\{x_k\}$ such that the values

$$\{\alpha f_1(x_k) - f_2(x_k\} \text{ tend to } -\infty .$$

This can happen in two ways. In the first case, $\{f_2(x_k)\}$ tends to $-\infty$ or $+\infty$.

Because of the way $\alpha$ was selected, this can happen only if

$$0 < \eta_1 = \eta_2 < \infty .$$

That means there is only one possible value for $\alpha$. For this value, there is a sequence of points for which the desired inequality fails no matter what the value of $\beta$ is. The other case is if the sequence of values $\{f_2(x_k)\}$ tends to a finite limit, and $\{f_1(x_k)\}$ tends to minus infinity. If such sequence exists, the infimum of the above problem would equal $-\infty$ for any value of $\alpha > 0$. QED.

It is interesting to see what happens in the linear case where the method reduces to the eliminating variable technique. Consider the example where

$$f_1(x) = x, \text{ and } f_2(x) = 5x .$$

Now

$$\eta_1 = 1/5 = \liminf x/(5x) \quad (\text{as } 5x \rightarrow +\infty), \text{ and}$$

$$\eta_2 = 1/5 = \limsup x/(5x) \quad (\text{as } 5x \rightarrow -\infty) .$$

The only possible value of $\alpha$ is therefore in $\{(5+5)/2\}$.

The problem $\{\inf(5)(x) - 5x\}$ has the infimum 0. Thus $\beta = 0$. We refer the reader here to Section 4.5 for an application of this.

Implementing Part One requires knowledge of the functions involved and sometimes, l'Hôpital's rule. Implementing Part Two can sometimes use the first and second order optimality conditions (abbreviated FONC, SONC, and SOSC) when the functions are twice continuously differentiable (See McCormick [27 chapters 10 and 11]). For some functions bounds can be obtained without these assumptions. A simple example is given here to illustrate the method. A more detailed example is given in Section 2.3.

Let $f_1(x) = x^4$ and $f_2(x) = x^2$. From Part One,

$$\liminf \{x^4/x^2\} = +\infty .$$

This implies that $\alpha$ can be any number $>0$. Set $\alpha = 1$. Part Two requires the minimization of

$$x^4 - x^2$$

over the real line. Using the FONC implies that minimizers might be 0 or $\pm\sqrt{1/2}$. Use of the SONC rules out zero. The SOSC imply that the others are strict local minimizers. Note that because $\alpha$ was chosen to taken care of the situation where local minimizers might be at $-\infty$ or $+\infty$, this simple analysis takes care of all cases. The value of $\beta$ is therefore $1/4$. The end result is that

$$x^4 + 1/4 \geqslant x^2 \text{ for all } -\infty < x < +\infty .$$

*The second method* for obtaining $\alpha$ and $\beta$ involves rewriting the inequality

$$\alpha f_1(x) + \beta \geqslant f_2(x)$$

as

$$\alpha f_1(x) - f_2(x) \geqslant -\beta\ , \tag{2.1.1}$$

or

$$\alpha \geqslant [f_2(x) - \beta]/f_1(x) \quad \text{when } f_1(x) > 0 \text{ for all } x\ . \tag{2.1.2}$$

When $f_1(x)$ is not always positive, 2.1.2 can be rewritten as:

$$\alpha \geqslant [f_2(x) - \beta + M\alpha]/[f_1(x) + M]\ , \tag{2.1.3}$$

where $-M$ is lower bound on $f_1(x)$. If there is no lower bound on $f_1(x)$, then 2.1.1 must be used.

Finding $\alpha$ and $\beta$ such that the original inequality holds on the real line can be done using either 2.1.1 or 2.1.2. Using 2.1.1, one algorithm is to find a value of $\alpha > 0$ such that the problem:

$$\inf_x \{\alpha f_1(x) - f_2(x)\}$$

has a finite minimum. Call it $v^*(\alpha)$. Setting $\beta(\alpha) = -v^*(\alpha)$ gives a pair satisfying the original inequality.

Using 2.1.3, the implied algorithm is to find a value of $\tilde{\beta}$ such that:

$$\sup_x \{[f_2(x) - \tilde{\beta}]/[f_1(x) + M]\}$$

has a maximum greater than zero. Call this $v^*(\tilde{\beta})$. Setting $\alpha(\tilde{\beta}) = v^*(\tilde{\beta})$ and $\beta = +v^*(\tilde{\beta})M + \tilde{\beta}$ also yields a pair satisfying the original inequality for all $-\infty < x < \infty$.

In Hamed and McCormick [9] are many example of these two approaches.

## 3. Finding Bounds on Single Variables

The material in the previous section will be used in the development of techniques for reducing systems of inequalities to inequality constraints involving functions of a single variable in Section 4. In this section the problem addressed is: given a set of inequality constraints of the form

$$f_k(x) \leqslant b_k\ , \quad k = 1, \ldots, p$$

on the single variable $x$, find a set of bounds on the variable $x$ itself.

The applicability of this to finding bounds on all the variables is shown in Section 4.5 where Fourier's method for linear inequality constraints is generalized to nonlinear inequalities.

Techniques for finding the bounds on $x$ depend a great deal on the nature of the functions $\{f_k(x)\}$. There are lots of cases and many are discussed in Hamed and McCormick [9] and in Hamed [8]. The ability to do this is essential to the generalized method of Fourier. We give here only one example of how to do this for the unary function $\sin(x)$.

In the set $\{x: -\pi \leq x \leq +\pi\}$ there are at most three intervals whose union is the set of points satisfying $-1 \leq a \leq \sin(x) \leq b \leq 1$. Call these $\{[L_j, U_j]\}$.
Let $I = \{i = 0, \pm 1, \pm 2, \ldots\}$. On the real line the set of points satisfying the inequality are all points in the set

$$\bigcup_{i \in I} \bigcup_{j=1,2,3} \{x: 2\pi i + L_j \leq x \leq 2\pi i + U_j\}\,.$$

## 4. Finding Bounds Pursuing Fourier's Ideas

This section is a generalization of the work in Fourier [6] for finding bounds on variables satisfying linear inequality constraints. An excellent exposition of Fourier's work is in Williams [45]. In Section 4.1 is a brief description of this method.

In Section 4.5 the full algorithm is stated. Three subalgorithms required are outlined in Sections 4.2, 4.3 and 4.4. In Section 4.2 a way is given of replacing certain nonlinear convex inequality constraints by linear inequality constraints. Conditions are proved under which the boundedness of the feasible region is preserved.

In Section 4.3 ad hoc techniques for developing functional lower and upper bounds on a single variable are discussed in general, with some examples. In Section 4.4 is given a general method for converting systems of nonseparable constraints to equivalent systems of separable inequality constraints.

Section 4.5 provides the generalized Fourier method for taking a system of separable nonlinear inequality constraints and finding bounds on the variables implied by these constraints. Section 4.6 contains examples of the generalized Fourier method.

### 4.1. FOURIER'S METHOD FOR LINEAR INEQUALITIES

In this section we briefly summarize Williams [45] work. For simplicity consider the set of linear inequality constraints

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i\,, \quad i = 1, \ldots, p\,.$$

Let $I$ be the index of a constraint where $a_{In} < 0$, and $K$ the index of a constraint where $a_{Kn} > 0$. (If either an $I$ or a $K$ does not exist, the feasible region is unbounded). Then the following chain of inequalities results:

$$\left(\sum_{j=1}^{n-1} - a_{Ij} x_j + b_I\right) / a_{In} \leq x_n \leq \left(\sum_{j=1}^{n-1} - a_{Kj} x_j + b_K\right) / a_{Kn}\,.$$

A necessary and sufficient condition that a point $(x_1, \ldots, x_n)$ exists satisfying the $I$th and $K$th inequalities is that a point $(x_1, \ldots, x_{n-1})$ exists satisfying

$$\sum_{j=1}^{n-1} [-a_{Ij}/a_{In} + a_{Kj}/a_{Kn}] x_j + b_I/a_{In} - b_K/a_{Kn} \leq 0\,.$$

Taking all pairs of the original inequalities in which $x_n$ appears in the same manner yields another set of inequalities in only $n-1$ variables. It is not difficult to show that the set defined by this new enlarged set of inequalities in $n-1$ variables is equal to the projection of the original set on $R^{n-1}$. Proceeding inductively it is clear that eventually $x_1$ appears alone in a system of bounds. If these are inconsistent, there is no point satisfying the original set; if there is no finite lower bound, and/or there is no finite upper bound, the set of points satisfying the original set of inequalities is unbounded. If a feasible value of $x_1$ is available, a backwards inductive procedure gives successive vectors $(x_1, x_2) \ldots (x_1, \ldots, x_n)$ satisfying the inductively generated inequality constraints. Ultimately a point satisfying the original set of inequality constraints is obtained. If the set of points satisfying the original inequalities is nonempty and bounded, then finite upper and lower bounds are obtained for each variable.

### 4.2. REPLACING CONVEX NONLINEAR INEQUALITY CONSTRAINTS BY LINEAR INEQUALITY CONSTRAINTS

In certain circumstances it is possible to bound the set of points determined by a set of nonlinear inequality constraints of the form

$$f_i(x) \leq b_i\,, \quad i=1,\ldots,m \quad (x \in E^n)\,,$$

where the $\{f_i(x)\}$ are convex differentiable functions by a set of linear inequality constraints, and *maintain the boundedness of the feasible region**. A basic tool is the well-known convex differential inequality:

$$f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0) \tag{4.2.0}$$

which holds when $f(x)$ is a convex $C^1$ function.
In anticipating the application of this technique to the generalized Fourier method, the maintenance of a bounded region is an essential property. In the following, a circumstance under which this occurs is established.

Suppose a set of inequality constraints has the form

$$f_i(x) \leq b_i\,, \quad i=1,\ldots,m-1 \tag{4.2.1}$$

and

$$f_m(x) = a_0^T x + \sum_{k=1}^{p} U_k(a_k^T x) \leq b \tag{4.2.2}$$

where the $U_k$'s are convex $C^1$ functions of a single argument. The derivative of $U_k(u)$ is denoted by $U_k'(u)$.

The Convex Replacement Algorithm which attempts to substitute a system of convex inequalities will be given later. A subalgorithm needed to replace one nonlinear term at a time will now be given.

We will eliminate the nonlinear term $U_1(a_1^T x)$ by replacing $f_m(x) \leq b$ by two

*It is assumed throughout that these inequalities define a nonempty set.

inequality constraints. Specifically, replace (4.2.2) with

$$a_0^Tx + U_1(M) + U_1'(M)(a_1^Tx - M) + \sum_{k=2}^{p} U_k(a_k^Tx) \leq b\ , \tag{4.2.3}$$

and

$$a_0^Tx + U_1(-M) + U_1'(-M)(a_1^Tx + M) + \sum_{k=2}^{p} U_k(a_k^Tx) \leq b\ , \tag{4.2.4}$$

where $M$ is some number greater than zero. We shall show that if $M$ is large enough, the boundedness of the original feasible region implies the boundedness of the 'linearized' region. Note that any point satisfying (4.2.1.) and (4.2.2) also satisfies (4.2.2), (4.2.3) and (4.2.4). In the theorem we assume such points exist.

THEOREM 4.1. *If the set of points satisfying (4.2.1) and (4.2.2) is a nonempty bounded set, then for $M$ large the set of points satisfying (4.2.1.), (4.2.3) and (4.2.4) is a nonempty bounded set.*

*Proof.* Assume the contrary. Then for every $M > 0$ there is a point $x\#$ satisfying (4.2.1) and (4.2.2) and a direction vector $s_M$, with $\|s_M\| = 1$ such that $x\# + s_Mt$ satisfies (4.2.3) and (4.2.4) for all $t$, but not (4.2.1) and (4.2.2).

Using (4.2.0), the two inequality constraints result:

$$\begin{aligned} b \geq\ & a_0^Tx\# + U_1(M) + U_1'(M)(a_1^Tx\# - M) + [a_0^Ts_M + U_1'(M)(a_1^Ts_M)]t \\ & + \sum_{k=2}^{p} [U_k(a_k^Tx\# + a_k^Ts_M\tau) + U'(a_k^Tx\# + a_k^Ts_M\tau)^T(a_ks_M)(t-\tau)]\ , \end{aligned} \tag{4.2.5}$$

and

$$\begin{aligned} b \geq\ & a_0^Tx\# + U_1(-M) + U_1'(-M)(a_1^Tx\# + M) + [a_0^Ts_M + U_1'(-M)(a_1^Ts_M)]t \\ & + \sum_{k=2}^{p} [U_k(a_k^Tx\# + a_k^Ts_M\tau) + U_k'(a_k^Tx\# + a_k^Ts_M\tau)(a_k^Ts_M)(t-\tau)]\ , \end{aligned} \tag{4.2.6}$$

for every $\tau > 0$.

The right hand sides of (4.2.5) and (4.2.6) cannot go to $+\infty$ as $t \to \infty$. Thus, collecting terms involving $t$, the two following inequalities must hold for all $\tau > 0$, and all $M > 0$:

$$a_0^Ts_M + U_1'(M)(a_1^Ts_M) + \sum_{k=2}^{p} U_k'(a_k^Tx\# + a_k^Ts_M\tau)(a_k^Ts_M) \leq 0\ , \tag{4.2.7}$$

and

$$a_0^Ts_M + U_1'(-M)(a_1^Ts_M) + \sum_{k=2}^{p} U_k'(a_k^Tx\# + a_k^Ts_M\tau)(a_k^Ts_M) \leq 0\ . \tag{4.2.8}$$

Because of the basic property of unary convex $C^1$ functions that their derivatives are nondecreasing it follows that if $a_k^Ts_M \geq 0$, then

$$\lim_{\tau \to \infty} U_k'(a_k^Tx\# + a_k^Ts_M\tau) > -\infty\ .$$

Also, if $a_k^T s_M \leq 0$,

$$\lim_{\tau\to\infty} U_k'(a_k^T x\# + a_k^T s_M \tau) < +\infty .$$

From these results and inequalities (4.2.7) and (4.2.8) it follows that

$$\lim_{\tau\to\infty} |U_k'(a_k^T x\# + a_k^T s_M \tau)| < \infty .$$

This fact will be used implicitly in the remainder of the proof.

Because $s_M$ is of length one there is at least one point of accumulation as $M \to \infty$. Call one such point $\bar{s}$. We shall show that the $x\# + \bar{s}t$ satisfies (4.2.1) and perturbation of (4.2.2), for all $t$.

Expanding $f_m(x\# + \bar{s}t)$, using Taylor's theorem, yields

$$f_m(x\# + \bar{s}t) = a_0^T x\# + a_0^T \bar{s}t$$
$$+ \sum_{k=1}^{p} (U_k(a_k^T x\#) + U_k'[a_k^T x\# + (a_k^T \bar{s})\tau(x\#, s, t)](a_k \bar{s})t\} , \tag{4.2.9}$$

where $\tau(x\#, s, t)$ is a number between 0 and $t$. This will be shortened to $\hat{\tau}$ for notational convenience.

We need to show that $f_m(x\# + \bar{s}t)$ is bounded above by a number independent of $t$. For this it is sufficient to show that for all $t > 0$:

$$a_0^T \bar{s} + U_1'(a_1^T x\# + a_1^T \bar{s}\hat{\tau})(a_1^T \bar{s}) + \sum_{k=2}^{p} U'(a_k^T x\# + a_k^T \bar{s}\hat{\tau})(a_k^T \bar{s}) \leq 0 . \tag{4.2.10}$$

There are several cases to consider.

CASE (i). $U_1'(M) \to +\infty$, and $U_1'(-M) \to -\infty$ as $M \to \infty$.

Dividing (4.2.7) by $U_1'(M)$ and (4.2.8) by $U_1'(-M)$ and taking the limit as $M \to \infty$ yields

$$a_1^T \bar{s} \leq 0 ,$$

and

$$a_1^T \bar{s} \geq 0 .$$

These imply:

$$a_1^T \bar{s} = 0 . \tag{4.2.11}$$

Because of the assumptions, for $M$ large it is possible to add positive multiples of (4.2.7) and (4.2.8) to yield

$$a_0^T s_M + \sum_{k=2}^{p} U_k'(a_k^T x\# + a_k^T s_M \tau)(a_k^T s_M) \leq 0 ,$$

for all $M > 0$, all $\tau > 0$. Taking the limit as $M \to \infty$,

$$a_0^T\bar{s} + \sum_{k=2}^{p} U_k'(a_k^T x\# + a_k^T \bar{s}\tau)(a_k^T \bar{s}) \leq 0\,, \quad \text{for all } \tau > 0\,.$$

Using $\tau = \hat{\tau}$ and (4.2.11), the above implies that (4.2.10) is true.

CASE (ii). $U_1'(M) \to \infty$, $U_1'(-M) \to \beta$, $|\beta| < \infty$, *as* $M \to \infty$.
Dividing (4.2.7) by $U_1'(M)$ and taking the limit yields

$$a_1^T \bar{s} \leq 0\,. \tag{4.2.12}$$

Taking the limit in (4.2.8) yields (for all $\tau > 0$)

$$a_0^T\bar{s} + \beta(a_1^T \bar{s}) + \sum_{k=2}^{p} U_k'(a_k^T x\# + a_k^T \bar{s}\tau)(a_k^T \bar{s}) \leq 0\,. \tag{4.2.13}$$

Because of (4.2.12) it follows that

$$U_1'(a_1 x\# + a_1 \bar{s}\tau) \geq \beta\,.$$

Then using (4.2.12) again it follows that

$$(a_1^T \bar{s}) U'(a_1^T x\# + a_1^T \bar{s}\tau) \leq (a_1^T \bar{s})\beta\,.$$

With this, the left hand side of (4.2.10) is $\leq$ the left hand side of (4.2.13) (evaluated at $\tau = \hat{\tau}$) and therefore the desired inequality follows.

CASE (iii). $U_1'(M) \to \alpha$, $|\alpha| < \infty$, $U(-M) \to -\infty$, as $M \to \infty$.
The proof is similar to that of Case (ii) and is omitted.

CASE (iv). $U_1'(M) \to \alpha$, $|\alpha| < \infty$, $U(-M) \to \beta$ , $|\beta| < \infty$ as $M \to \infty$.
The proof of this final case is also omitted.

It has been shown that for all possible behaviour of $U_1'$, (4.2.10) holds. From this it follows that $f_m(x\# + st)$ is bounded above by a bound independent of $t$.

Because the functions $f_i(x)$ $(i = 1, \ldots, m-1)$ are convex functions it follows that $f_i(x\# + \bar{s}t) \leq b_i$ for $i = 2, \ldots m-1$. By Theorem 54 in [5] it follows from the assumption that the nonempty region satisfying (4.2.1) and (4.2.2) is bounded that all points on the ray $x\# + \bar{s}t$ cannot satisfy a finite perturbation of these inequalities. This contradiction proves the theorem.

We note that continuous differentiability of the functions is not the key requirement for this method. The use of a subgradient at $M$ instead of $U'(M)$ would also suffice. The main requirement is the property that subgradients of unary convex functions are nondecreasing.

Suppose a set of inequality constraints has the form

$$a_{i0}^T x + \sum_{k=1}^{p_i} U_{ik}(a_{ik}^T x) \leq b_i\,, \quad \text{for } i = 1, \ldots, m\,, \tag{4.2.18}$$

where the $U_{ik}$'s are convex $C^1$ functions of a single argument. A general algorithm which replaces these with a set of linear inequality constraints is stated below.

*Convex Linear Replacement Algorithm*

The general idea is to use the convex differentiable inequality on all the nonlinear $U'_{ik}$'s defining the inequality constraints (4.2.18) as indicated in the previous development. There is one matter which must be addressed.

Applied recursively, the method generates $a_0$'s which depend upon the values of the derivatives used at the previous approximations. Thus we can generate successive bounded sets when $M$ is large only if the inequalities (4.2.7) and (4.2.8) have a vector $a_0$ which is of a smaller order than the $U'(M)$ which appears there. Dividing (4.2.7) by $U'(M)$ or (4.2.8) by $U'(-M)$ will not necessarily drive the term involving $a_0^T s_M$ to zero. For this reason, each time the subalgorithm is invoked, the $M$ used must take into account the previous ones and is a function of the iteration number.

*Iteration* 0. Denote by $L_0$ the set of inequality constraints (4.2.18). Let $M_0 = 1$, let $M$ be a preassigned number $>1$.

*Iteration* $l$, $l \geq 1$. Available is a set $L_{l-1}$ of inequality constraints, each having the form (4.2.2). If all the constraints are linear, the algorithm is done. Otherwise, pick some nonlinear term from one of the constraints. Without loss of generality suppose the unary convex term is $U_{IK}$.

If $l = 1$, set $M_1 = M$.

If $l > 1$, a more complication choice for $M_l$ is required.

The value $a_{l0}$ is a function of derivatives of the previous approximations, and by induction can be regarded as a function of $M$. Thus $a_{l0} = a_{l0}(M)$.

The value $M_l$ to be used depends upon the case involved. If $U_{IK}$ is of Case (i), $M_l = M_l(M)$ should be picked large enough so that

$$\|a_{l0}(M)\| / U'_{IK}[M_l(M)] \to 0\,, \tag{4.2.19}$$

and

$$a_{l0}(M) / U'_{IK}[-M_l(M)] \to 0\,, \tag{4.2.20}$$

as $M \to \infty$.

Case (ii) requires only that (4.2.19) hold, and Case (iii) requires only that (4.2.20) hold. Case (iv) only requires that $M_l$ go to infinity, so it is set equal to $M$.

After the selection of $M_l$, the convex replacement subalgorithm is to be used on the term $U_{IK}$.

THEOREM 4.2. *If M is sufficiently large, and if the set of points feasible to*

*(4.2.18) is nonempty and bounded, then the resulting set of linear inequalities produced by the above algorithm is a bounded nonempty set.*

*Proof.* The proof follows inductively from Theorem 4.1 and the properties stated above which are required to hold for the $M_l$'s.

It is worth mentioning here that there is a trade-off in choosing $M$. One can speed up the algorithm by starting with a very big $M$. The cost of doing this is having a very loose bound. In fact there is a specific $M$ that gives the most tight bound but it is hard to determine.

There are two algorithms derived from the above which can be used to find bounds for a set given by the convex inequality constraints (4.2.18).

The first algorithm is to generate a sequence of unbounded values $\{M(q)\}$. At the $q$th iteration generate the linear inequality constraints using the convex replacement algorithm with $M = M(q)$. Then use Fourier's method for finding bounds as applied to this set of linear inequalities. If Fourier's method indicates that the set satisfying the linear inequalities is not bounded the procedure is initiated again for $M(q+1)$. At iteration $k$, we use $M$ equal to $M^{1+\epsilon k}$, thus

$$\lim_{k\to\infty} \{M^{1+\epsilon(k-1)}/M^{1+\epsilon k}\} \to 0\,.$$

If the original set of inequalities is bounded, eventually the set of linear inequalities will be bounded and Fourier's original method will find some bounds.

A second method is to generate all the coefficients symbolically as a function of $M$. Fourier's method would be applied to the symbolic coefficients and bounds would be found in terms of $M$. A search on $M$ would then be made to find if for large values, the variables are bounded.

*Nonconvex Case*

The desirable property of inherited boundedness can sometimes apply in the convex case as indicated above. But it is not true for the nonconvex case as seen in the following example. Thus the technique suggested above is just a heuristic one for the general case.

EXAMPLE 4.2.3. Consider the set of inequality constraints. $g_1 = x^2 \leqslant 1 + y^2$, $g_2 = x - 2y \leqslant 1$, and $g_3 = -x + 3y \leqslant 1$. It will be shown in Section 4.6 that this defines a bounded set of points.

We will expand $x^2$ around $M$ and $-M$. Thus the first inequality above is replaced by the two constraints

$$L_1 = 1 + y^2 + M^2 - 2Mx \geqslant 0\,,$$

and

$$L_2 = 1 + y^2 + M^2 + 2Mx \geq 0 .$$

We will show that eventually points along the line $x - 2y = 1$ (as $x \to -\infty$) satisfy these inequalities (as well as $-x + 3y \leq 1$). Clearly the first inequality above is satisfied for any $x < 0$. Make the substitution $x = 2y + 1$ in the second inequality. This gives the one dimensional inequality

$$L_3 = y^2 + 4My + M^2 + 2M + 1 \geq 0 .$$

Clearly no matter how large $M$ is, as $y \to -\infty$, this inequality will be satisfied. Thus for all $y$ small, the point $(2y + 1, y)$ satisfies the four inequalities generated by replacing the original nonlinear inequality by two constraints using the differential inequality.

### 4.3. AD HOC METHODS FOR FINDING A FUNCTIONAL BOUND ON A SINGLE VARIABLE

In this section we explore *ad hoc* methods for generating from an inequality of the form

$$h(x_J) \leq \sum_{\substack{j=1 \\ j \neq J}}^{n} g_j(x_j) + b$$

inequalities of the form

$$\sum_{\substack{j=1 \\ j \neq J}}^{n} g_j^L(x_j) + b^L \leq x_J \leq \sum_{\substack{j=1 \\ j \neq J}}^{n} g_j^U(x_j) + b^U .$$

Our aim is to transform a functional bound on a function of the single variable $x_J$ into functional lower and/or upper functional bounds on the single variable itself; with the stipulation that the bounding function forms be *separable* in the other variables. There seems to be no general algorithm which can be stated, but only special cases cited for which this is possible.

This is often done through inverse functions. Three examples of useful inequalities are given below.

$$\left(\sum_{k=1}^{p} a_k\right)^{0.5} \leq \sum_{k=1}^{p} |a_k|^{0.5} .$$

$$\text{LN}\left(\sum_{k=1}^{p} a_k\right) \geq \left[\sum_{k=1}^{p} \text{LN}(a_k)\right] / p , \quad \text{when } a_k > 0 , \quad k = 1, \ldots, p .$$

$$\text{EXP}\left(\sum_{k=1}^{p} a_k\right) \leq \sum_{k=1}^{p} \text{EXP}(pa_k) .$$

EXAMPLE 4.3.1. $h(x_1) = (x_1)^2$.

It can be easily shown that

$$\sum_{j=2}^{n} -|g_j(x_j)|^{0.5} - |b|^{0.5} \leq x_1 \leq \sum_{j=2}^{n} |g_j(x_j)|^{0.5} + |b|^{0.5} .$$

The use of this in finding bounds is illustrated by the example in Section 5.1. The set of inequality constraints is: $x^2 \leq y^2 + 1$, $x - 2y \leq 1$, $-x + 3y \leq 1$. Using the general result above, the nonlinear inequality yields the two inequality constraints:

$$-|y| - 1 \leq x \leq |y| + 1 .$$

It will be shown rigorously in Section 4.6 that the resulting region (using the two given linear inequalities above) is bounded.

## 4.4. CONVERTING SYSTEMS OF NONSEPARABLE CONSTRAINTS TO EQUIVALENT SYSTEMS OF SEPARABLE CONSTRAINTS

A function $f(x)$ is a *separable* function of the $n$ variables $(x_1, \ldots, x_n)$ if it has the form

$$f(x) = \sum_{j=1}^{n} f_j(x_j) .$$

In this section we give a way of converting a system of inequality constraints of the form

$$g_i(x) \leq b_i , \quad i = 1, \ldots, m \tag{4.4.1}$$

to an 'equivalent' set of inequality constraints involving more variables and more constraints. Suppose the set of variables in the separated set is $(x, y)$, where $y \in E^q$, and the separated set of inequality constraints is

$$\sum_{j=1}^{n} G_{ij}(x_j) + \sum_{k=1}^{q} H_{ik}(y_k) \leq d_i , \quad i = 1, \ldots, M . \tag{4.4.2}$$

The two sets are considered *equivalent* if the following is true:

(i) if $x$ satisfies (4.4.1), then there is a $y$ such that $(x, y)$ satisfies (4.4.2), and
(ii) if $(x, y)$ satisfies (4.4.2), then $x$ satisfies (4.4.1).

It is not known how to separate all problems. We assume here that the functional forms defining (4.4.1) are not separable because they have product terms and/or they involve unary functions of more than one variable. The process to separate such problems is well-known. A discussion of it is contained in McCormick [25, 28]. We note the obvious fact that an equality constraint can be written as two inequality constraints. The process of separation involves the recursive use of the two steps A and B below.

*Step A.* If a cause of nonseparability in a system of inequality constraints is the existence of a term of the form $T[t(x)]$ where $T(t)$ is a unary function and $t(x)$ is a function of more than one variable; introduce a new variable $y$, and equality constraint $y = t(x)$. Replace $T[t(x)]$ by $T(y)$ wherever it occurs.

*Step B.* If a cause of nonseparability in a system of inequality constraints is a product term of the form $u(x) * v(x)$, introduce two new variables $w$, $z$ and two equality constraints $w + z = u(x)$, and $w - z = v(x)$. Replace $u(x) * v(x)$ by $w^2 - z^2$ wherever it occurs.

It is easy to show that the sets of inequality constraints generated by these steps are equivalent (see McCormick [25] for a proof of this).

Systems of inequalities involving factorable functions can be separated [28]. An example of a function which to date has not been able to be separated is the gamma distribution function:

$$\int_0^a \{[(x_1x_2)^{x_2}t^{(x_2-1)}e^{-x_1x_2t}]/\Gamma(x_2)\}\,dt$$

(see McCormick [27, p. 83]).

### 4.5. THE GENERALIZED FOURIER METHOD

The basic idea behind the generalized Fourier method is tot take a system of nonseparable inequality constraints and convert it to a system of separable inequality constraints. Techniques are then applied to this latter set which hopefully generate lower and upper bounds on the variables. If bounds are found, they are valid ones for the variables satisfying the original set of inequality constraints. Except in special cases however, there is no guarantee that the separated system of inequality constraints defines a bounded set. The algorithm requires three separate processes.

*Process One* (creation of a separable set of constraints).

The set of separated inequality constraints consists of three parts: set A, set B, and set C.

Set A is obtained by using the techniques of Section 4.4 on the set of original inequality constraints. This will, in general, have more variables and constraints than the original set.

Next, apply the techniques of Section 4.2 to the convex unary terms of the *original* set of inequality constraints. This creates a set of partially linearized constraints. Apply the separation techniques of Section 4.4 to this partially linearized set. This results in the set B.

Next apply the *ad hoc* techniques of Section 4.3 to the union of sets A and B.

The constraints resulting from these constitute set C. The union of sets A, B and C is the input to the second process. Call this set $S_1$.

*Process Two* (reduction of separable constraints to constraints in only one variable).

The techniques of Section 2 are applied to set $S_1$ to attempt to 'eliminate' the variable $x_1$. If this is sucessful, this results in a set $S_2$ which contains separable constraints in the variables $(x_2, \dots, x_N)$.

In general, at the $p$th iteration of the second process, there is a set of separable inequality constraints $S_p$ which contains only the variables $(x_p, \dots, x_N)$. The techniques of Section 2 are applied to this set to eliminate $x_p$. If successful, a set $S_{p+1}$ results which contains only the variables $(x_{p+1}, \dots, x_N)$. Process Two terminates when the set $S_N$ has been created.

To show specifically how the techniques of Section 2 apply, consider without loss of generality that we are at the beginning of the $p$th step in the above process. Available are $m_p$ constraints of the form:

$$g_{ip}(x_p) \leq \sum_{j=p+1}^{N} -g_{ij}(x_j) + b_i\,, \quad i = 1, \dots, m_p\,.$$

Suppose for two constraints, the $K$th and $I$th there exists a scalar $\alpha > 0$, and a scalar $\beta$ such that:

$$0 \leq g_{Ip}(x_p) + \alpha g_{Kp}(x_p) + \beta\,, \quad \text{for } -\infty < x_p < \infty\,.$$

The inequalities obtain that:

$$\sum_{j=p+1}^{N} g_{Ij}(x_j) - b_I \leq -g_{Ip}(x_p) \leq \alpha g_{Kp}(x_p) + \beta$$

$$\leq \alpha\left[\sum_{j=p+1}^{N} -g_{Kj}(x_j) + b_K\right] + \beta\,.$$

Combining the first and last terms yields an inequality which is separable in the variables $(x_{p+1}, \dots, x_N)$.

*Process Three* (finding bounds on $x_N, \dots, x_1$).

The set $S_N$ has constraints only on the single variable $x_N$. The techniques of Section are used to find a lower and upper bound on $x_N$.

If this is successful, denote these bounds by $L_N$ and $U_N$. These bounds with the set of constraints $S_{N-1}$ can be converted, using standard techniques of interval arithmetic [29], to a set of constraints in only the variable $x_{N-1}$. Again, the techniques of Section 3 are used to find a lower and an upper bound on $x_{N-1}$.

This backward process is repeated: i.e., the bounds $\{L_i, U_i\}$, $i = p+1, \dots, N$ are used with techniques of interval arithmetic to create constraints on the variable $x_p$. The techniques of Section 3 are used on these constraints to create a lower and upper bound on $x_p$. If each inductive step is successful, the final result is a set of bounds on all the variables.

More specifically, assume that the bounds $\{L_i, U_i\}$ for $i = p+1, \ldots, N$ are available. Set $S_p$ which was constructed during Process Two contains constraints:

$$g_{ip}(x_p) \leq \sum_{j=p+1}^{N} - g_{ij}(x_p) + b_i\,, \quad i = 1, \ldots, m_p\,.$$

Techniques of interval arithmetic can be applied to yield upper bounds on the right hand sides.

Then the techniques of Section 3 can be applied to find a lower and upper bound on $x_p$.

### 4.6. EXAMPLES OF THE USE OF THE GENERALIZED FOURIER METHOD

EXAMPLE 4.6.1. Consider the constraints

$$g_1 = x^2 \leq y^2 + 1\,, \quad g_2 = x - 2y \leq 1\,, \quad \text{and} \quad g_3 = -x + 3y \leq 1\,.$$

Using the inequality from example 4.3.1. results in

$$|x| \leq |y| + 1 \tag{a}$$

$$x \leq 2y + 1 \tag{b}$$

and

$$-x \leq -3y + 1\,. \tag{c}$$

Adding (a) and (b) yields $0 \leq |y| + 2y + 2$ which implies $y \geq -2$. Adding (a) and (c) yields $0 \leq |y| - 3y + 2$ which implies $y \leq 1$. Adding (b) and (c) also implies $y \leq 1$. Substituting the bounds on $y$ back into the original constraints implies $-\sqrt{5} \leq x \leq \sqrt{5}$.

EXAMPLE 4.6.2. The inequality constraints are

$$x_2 + (x_1 + x_3)^2 + 2x_3 \leq 19\,, \tag{a}$$

$$-x_2 + (x_1 + x_2)^3 \leq 8\,, \tag{b}$$

and

$$2x_1 - 2x_2 + x_3 \leq 20\,. \tag{c}$$

Using the convex differential inequality on (a), at

$$x_1 + x_3 = 4\,, \quad \text{and} \quad x_1 + x_3 = -4 \text{ gives}$$

$$8x_1 - 2x_2 + 10x_3 \leq 35 \tag{d}$$

and

$$-8x_1 + x_2 - 6x_3 \leq 35\,. \tag{e}$$

Define $x_4 = x_1 + x_2$. Then (b) is equivalent to

$$-x_2 + x_4^3 \leqslant 8\,, \qquad \text{(f)}$$

$$-x_1 - x_2 + x_4 \leqslant 0\,, \qquad \text{(g)}$$

and

$$x_1 + x_2 - x_4 \leqslant 0\,. \qquad \text{(h)}$$

We now eliminate $x_1$.

| | | | |
|---|---|---|---|
| 4(c) + (e) | $\Rightarrow -7x_2 - 2x_3$ | $\leqslant 115$ | (i) |
| (d) + (e) | $\Rightarrow 2x_2 + 4x_3$ | $\leqslant 70$ | (j) |
| (c) + 2(g) | $\Rightarrow -4x_2 + x_3 + 2x_4$ | $\leqslant 20$ | (k) |
| (d) + 8(g) | $\Rightarrow -7x_2 + 10x_3 + 8x_4$ | $\leqslant 35$ | (l) |
| (e) + 8(h) | $\Rightarrow 9x_2 - 6x_3 - x_4$ | $\leqslant 35$ | (m) |
| (f) | $\Rightarrow -x_2 + x_4^3$ | $\leqslant 8$ | (n) |

We now eliminate $x_2$.

| | | | |
|---|---|---|---|
| 2(i) + 7(j) | $\Rightarrow 24x_3$ | $\leqslant 720$ | (o) |
| 9(i) + 7(m) | $\Rightarrow -60x_3 - 7x_4$ | $\leqslant 1280$ | (p) |
| 2(j) + k | $\Rightarrow 5x_3 + 2x_4$ | $\leqslant 160$ | (q) |
| 7(j) + 2(l) | $\Rightarrow 48x_3 + 16x_4$ | $\leqslant 560$ | (r) |
| (j) + 2(n) | $\Rightarrow 4x_3 + 2x_4^3$ | $\leqslant 86$ | (s) |
| 9(k) + 4(m) | $\Rightarrow -15x_3 + 14x_4$ | $\leqslant 176$ | (t) |
| 9(l) + 7(m) | $\Rightarrow 48x_3 + 65x_4$ | $\leqslant 560$ | (u) |
| m + 9(n) | $\Rightarrow -6x_3 - x_4 + 9x_4^3$ | $\leqslant 107$ | (v) |

In this example it is not necessary to eliminate ($x_3$).

from (0): $x_3 \leqslant 30\,,$

from (p): $-7x_4 \leqslant 1280 + 60(30) \Rightarrow x_4 \geqslant -440\,,$

from (t): $176 \geqslant -15x_3 + 14x_4 \geqslant -15x_3 + (14)(-440) \Rightarrow x_3 \geqslant -422.4\,,$

from (q): $160 \geqslant 5x_3 + 2x_4 \geqslant (5)(-422.4) + 2x_4 \Rightarrow x_4 \leqslant 1{,}136.00\,,$

from (i): $-7x_2 \leqslant 2(30) + 115 \Rightarrow x_2 \geqslant -25\,,$

from (j): $2x_2 \leqslant 4(422.4) + 70 \Rightarrow x_2 \leqslant 879.8\,,$

from (c): $2x_1 \leqslant 2(879.8) + 422.4 + 20 \Rightarrow x_1 \leqslant 1{,}001.0\,,$

from (e): $-8x_1 \leqslant 25 + 6(0) + 35 \Rightarrow x_1 \geqslant -4.375\,.$

REMARKS from (r): $560 \geq 48x_3 + 16x_4 \Rightarrow 560 \geq 48(-422.4) + 16x_4 \Rightarrow x_4 \leq 1{,}302.2$, from (u): $560 \geq 48x_3 + 65x_4 \Rightarrow 560 \geq 48(-422.4) + 65x_4 \Rightarrow x_4 \leq 300$, but from (q) we found that $x_4 \leq 1{,}136.00$. This demonstrates that we can get different bounds by different back substitutions. According to Fourier one should exhaust the combination of the given set of inequalities to find all possible bounds then choose the tightest one.

## 5. Summary and Conclusions

A motivation for this paper was to provide bounds on problem variables, since a key requirement of many of the most powerful existing techniques for solving nonconvex problems is that bounds on problem variables be available.

In this paper, different methods for obtaining lower and/or upper bounds on the problem variables are developed in order to enhance the methodology for solving nonconvex programming problems. Fourier in 1826 established a method for manipulating linear inequalities to obtain bounds on variables. This method is generalized in Section 4.5 to find bounds on variables satisfying nonlinear inequality constraints. The idea we present is to reduce the systems of separable nonlinear inequality constraints to inequalities involving a single variable. An inductive method obtains bounds on all the other variables. This requires the inequality constraints to be separable. A method of converting the certain large classes of nonlinear inequalities to an equivalent set in which the functions are separable was presented in McCormick [28]. Examples were provided to demonstrate the use of this approach.

Several techniques to find bounds on a single variable from inequality constraints are introduced in Section 3. Section 2 as well, characterizes functions and situations in which one can analyze the global behavior of the sum of two or more functions of the same variable. The general problem considered here is stated in the following simple form: given two functions $f_1(x)$ and $f_2(x)$ of a single variable $x$, find a positive scalar $\alpha$ and a scalar $\beta$ such that $\alpha f_1(x) + \beta \geq f_2(x)$ for $-\infty < x < +\infty$. Two different methods are devised to do this. The application for these techniques introduced in Sections 2 and 3 are given in Section 4.5 in the generalization of Fourier's idea.

Fourier's techniques [6] for manipulating linear inequalities to obtain bounds on variables is generalized in Section 4 for the case of nonlinear inequalities. Three subalgorithms required are outlined. These include: (1) a way of replacing certain nonlinear convex inequality constraints by linear inequality constraints, (2) ad hoc methods for finding a functional bound on a single variable, and (3) a general method for converting systems of nonseparable constraints to equivalent systems of separable inequality constraints. Those three subalgorithms lead to the generalized Fourier method for taking a system of separable nonlinear inequality constraints and finding bounds on the variables implied by these constraints. Examples were provided to demonstrate the use of this technique.

A natural extension of this paper would be the investigation of the use of the inverse function techniques on functions of more than one variable in order to eliminate variables and obtain bounds on the function variables.

## Note

It is worth mentioning that many papers presented at the ORSA/TIMS meeting in Philadelphia on 29–31 October 1990 [see ORSA/TIMS Bulletin Number 30] needed or assumed the existence of bounds on problem variables and/or objective function value. We cite here according to session number some of these papers: Amarger [MC30.4], Chang [TB30.4], Fisher [TA3.3], Keller [TB30.1], Konno [MB30.2], Lazimy [MC30.1], Nielson [MD30.5], Savard [TA30.1], Yang [MB30.3], Zhu [MB30.1].

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